On Continuous Binary Linearoid Groups, and the Corresponding Differential Equations and Λ *Functions.*

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In a former paper we have shown that, corresponding to every group of the form

$$\eta_i = \sum_{k=1}^{n} \phi_{ik}(x;\, a_1, \ldots . a_r)\, y_k\,, \tag{1}$$

where the r parameters a_i are essential, there exists a system of differential equations of order r, whose general solutions are given by (1), if $y_1, \ldots . y_n$ form a fundamental system. The functions ϕ_{ik} were supposed to be uniform functions of x, and it was found that, if the parameters a_i were properly chosen, ϕ_{ik} were uniform functions of the parameters also.

We propose, in this paper, to discuss these groups, the corresponding differential equations, and their solutions for the case that $n = 2$.

It may, of course, happen that ϕ_{ik} are all independent of x, so that the group becomes linear and $r \leqq 4$. Thus, the linear group appears as a special case of the linearoid. By using Lie's types of linear groups, the types of linearoid groups could be more easily found than by the methods of this paper. But the methods employed here, and which are also all essentially due to Lie, are more elementary and furnish not only the *types* of all groups investigated, but these groups themselves. Moreover, a number of interesting formulæ and theorems are thus found which, even for linear groups, escape notice when the *type* method is used.

§1.—*One-parameter Groups. Case I.*

Let

$$Uf = (\psi_{11}(x)\, y_1 + \psi_{12}(x)\, y_2)\, \frac{\partial f}{\partial y_1} + (\psi_{21}(x)\, y_1 + \psi_{22}(x)\, y_2)\, \frac{\partial f}{\partial y_2} \tag{2}$$

be the infinitesimal transformation of the group. Its finite transformations are found by integrating the simultaneous system

$$\frac{d\eta_i}{dt} = \psi_{i1} y_1 + \psi_{i2} y_2, \qquad (i = 1,\ 2) \qquad (3)$$

with the initial conditions $\eta_i = y_i$ for $t = 0$.

Denoting, therefore, by ρ_1, ρ_2, the roots of the quadratic

$$|\psi_{ik} - \delta_{ik}\rho| = 0, \qquad (\delta_{ii} = 1,\ \delta_{ik} = 0,\ i \neq k), \quad (4)$$

and determining λ_1 and λ_2 from the equations

$$\left.\begin{aligned} \lambda_i(\psi_{11} - \rho_i) + \psi_{12} &= 0, \\ \lambda_i \psi_{21} + \psi_{22} - \rho_i &= 0, \end{aligned}\right\} \qquad (i = 1,\ 2) \qquad (5)$$

the finite equations of the group are found to be

$$\left.\begin{aligned} \eta_1 &= \frac{1}{\lambda_1 - \lambda_2}(\lambda_1 e^{\rho_1 t} - \lambda_2 e^{\rho_2 t})\, y_1 - \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}(e^{\rho_1 t} - e^{\rho_2 t})\, y_2, \\ \eta_2 &= \frac{1}{\lambda_1 - \lambda_2}(e^{\rho_1 t} - e^{\rho_2 t})\, y_1 - \frac{1}{\lambda_1 - \lambda_2}(\lambda_2 e^{\rho_1 t} - \lambda_1 e^{\rho_2 t})\, y_2, \end{aligned}\right\} \qquad (6)$$

where λ_1 and λ_2, and, therefore, ρ_1 and ρ_2 must be distinct. We treat then under Case I, the case where the characteristic equation of the infinitesimal transformation of the group has distinct roots. The case of equal roots will be considered as Case II.

Our group is of the required form, i. e., its coefficients are uniform functions of x, if

$$D = (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} \qquad (7)$$

is the square of a uniform function of x.

The invariant of the group can be found without integration by noting the fact that there exist two relative invariants.

For from (6) we find easily

$$\left.\begin{aligned} \eta_1 - \lambda_1 \eta_2 &= e^{\rho_2 t}(y_1 - \lambda_1 y_2), \\ \eta_1 - \lambda_2 \eta_2 &= e^{\rho_1 t}(y_1 - \lambda_2 y_2). \end{aligned}\right\} \qquad (8)$$

If, then, we write

$$Y_i = y_1 - \lambda_i y_2, \quad H_i = \eta_1 - \lambda_i \eta_2, \qquad (i = 1,\ 2), \qquad (9)$$

Y_1, Y_2 are relative invariants, and

$$\vartheta_1 = Y_1^{\rho_1} Y_2^{-\rho_2} = H_1^{\rho_1} H_2^{-\rho_2} \tag{10}$$

is an absolute invariant.

We are interested in the differential invariants of the first order. According to (8), we have the following equations:

$$\frac{d}{dx}\frac{1}{\rho_j}\log H_i = \frac{d}{dx}\frac{1}{\rho_j}\log Y_i, \qquad (i, j = 1,\ 2;\ i \neq j).$$

We have, therefore, the differential invariants:

$$\vartheta_2 = \frac{Y_1'}{Y_1} - \frac{\rho_2'}{\rho_2}\log Y_1, \quad \vartheta_3 = \frac{Y_2'}{Y_2} - \frac{\rho_1'}{\rho_1}\log Y_2, \tag{11}$$

which are obviously independent. ϑ_1, ϑ_2, ϑ_3 are the three independent invariants of the original group G extended once. The relation holds:

$$\frac{1}{\vartheta_1}\frac{d\vartheta_1}{dx} = \frac{1}{\rho_1\rho_2}\frac{d(\rho_1\rho_2)}{dx}\log\vartheta_1 + \rho_1\vartheta_2 - \rho_2\vartheta_3. \tag{12}$$

Let us consider a system of differential equations

$$\Phi_k(\vartheta_1, \vartheta_2, \vartheta_3) = 0, \qquad (k = 1,\ 2,\ 3),$$

where Φ_k are three independent functions of ϑ_1, ϑ_2, ϑ_3 with coefficients depending only upon x, but such that if, by solving these equations, we find

$$\vartheta_k = f_k(x), \qquad (k = 1,\ 2,\ 3),$$

the equation

$$\frac{1}{f_1}\frac{df_1}{dx} = \frac{1}{\rho_1\rho_2}\frac{d(\rho_1\rho_2)}{dx}\log f_1 + \rho_1 f_2 - \rho_2 f_3 \tag{13}$$

will be identically satisfied. Then this system, or in full,

$$\left.\begin{aligned} \frac{d\log}{dx}(y_1 - \lambda_1 y_2) - \frac{d\log\rho_2}{dx}\log(y_1 - \lambda_1 y_2) &= f_2(x), \\ \frac{d\log}{dx}(y_1 - \lambda_2 y_2) - \frac{d\log\rho_1}{dx}\log(y_1 - \lambda_2 y_2) &= f_3(x), \\ (y_1 - \lambda_1 y_2)^{\rho_1}(y_1 - \lambda_2 y_2)^{-\rho_2} &= f_1(x), \end{aligned}\right\} \tag{14}$$

where (13) also is verified is to be considered.

However, in place of the last equation (14), it is often better to write

$$\rho_1 \log(y_1 - \lambda_1 y_2) - \rho_2 \log(y_1 - \lambda_2 y_2) = \log f_1(x) = f(x), \quad f_1(x) = e^{f(x)}. \quad (15)$$

For the left member of this equation is a more characteristic invariant of our group than $Y_1^{\rho_1} Y_2^{-\rho_2} = \vartheta_1$. For instance, ϑ_1 is invariant not only for the transformations (8) or

$$H_1 = e^{\rho_2 t}\, Y_1, \quad H_2 = e^{\rho_1 t}\, Y_2,$$

but also for all transformations of the form

$$H_1 = e^{\sigma_2 t_k} Y_1, \quad H_2 = e^{\sigma_1 t_k}\, Y_2,$$

provided that the relation is possible

$$(\rho_1 \sigma_2 - \rho_2 \sigma_1)\, t_k = 2k\pi i,$$

where k is any integer. Thus, ϑ_1 may admit, besides all of the transformations of our group, an infinite but discreet number of substitutions, i. e., ϑ_1 may admit altogether a mixed group.

If, now, y_1, y_2 is a fundamental system of our equations, the general solutions η_1, η_2 will be given by equations (6), where t is an arbitrary constant. If λ_1, λ_2; ρ_1, ρ_2; f_2; f_3; f_1 or f are uniform functions of x, the functions y_1, y_2 will have the following property: They are uniform, finite and continuous, except for certain values of x, say $a_1, a_2, \ldots$, and whenever the variable x describes a circuit around a_i, y_1, y_2 undergo a substitution A_i contained in the group (6) or some other substitution, leaving the left members of (14) unaltered, provided that there exists such a substitution. We shall leave such exceptional cases aside and prove only that the quantities still at our disposal can be so chosen that the points a_i and the substitutions A_i belonging to them can be arbitrarily assigned, all of the substitutions A_i being of form (6).

If y_1, y_2 are functions of the character just described, obviously f_1, f_2, f_3, f must be uniform functions of x. Let them, then, be uniform functions, and study directly the differential equations.

The integration of the first two equations (14) gives

$$\left.\begin{aligned} \log Y_1 &= \frac{\rho_2}{\rho_2^0}\left[C + \rho_2^0 \int_{x_0}^{x} \frac{f_2(x)}{\rho_2}\, dx\right], \\ \log Y_2 &= \frac{\rho_1}{\rho_1^0}\left[C' + \rho_1^0 \int_{x_0}^{x} \frac{f_3(x)}{\rho_1}\, dx\right], \end{aligned}\right\} \quad (16)$$

where ρ_i^0 denotes the value of ρ_i for $x = x_0$, an arbitrary value of x. But the two constants C and C' are not independent. For, according to the third equation (14), we must have

$$Y_1^{\rho_1} Y_2^{-\rho_2} = e^{\rho_1 \rho_2 \left(\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0}\right) + \rho_1 \rho_2 \int_{x_0}^{x} \left(\frac{f_2(x)}{\rho_2} - \frac{f_3(x)}{\rho_1}\right) dx} = f_1(x), \tag{17}$$

or, if we use (13), to simplify the exponential,

$$e^{\rho_1 \rho_2 \left(\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0}\right) + \log f_1(x) - \frac{\rho_1 \rho_2}{\rho_1^0 \rho_2^0} \log f_1(x_0)} = f_1(x).$$

Therefore,

$$\rho_1 \rho_2 \left[\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0} - \frac{1}{\rho_1^0 \rho_2^0} \log f_1(x_0)\right] = 2k\pi i$$

must be an integral multiple of $2\pi i$. If $\rho_1 \rho_2$ is not a constant, we conclude at once

$$\frac{C}{\rho_2^0} - \frac{C'}{\rho_1^0} - \frac{1}{\rho_1^0 \rho_2^0} \log f_1(x_0) = 0, \tag{18}$$

Let $a_1 \dots a_m$ be the poles of $\dfrac{f_2(x)}{\rho_2}$ and $\dfrac{f_3(x)}{\rho_1}$, so that

$$\frac{f_2(x)}{\rho_2} = \sum_{k=1}^{m} \frac{c_k}{x - a_k} + \dots, \quad \frac{f_3(x)}{\rho_1} = \sum_{k=1}^{m} \frac{d_k}{x - a_k} + \dots, \tag{19}$$

where the terms not written contain no terms of the form $\dfrac{1}{x - \lambda}$, and, therefore, give uniform functions on integration.

But, according to (13), we have

$$f_1(x) = e^{\rho_1 \rho_2 \left[\alpha + \int \left(\frac{f_2}{\rho_2} - \frac{f_3}{\rho_1}\right) dx\right]} \tag{20}$$

α being an arbitrary constant. But $f_1(x)$ must be a uniform function of x. If $\rho_1 \rho_2$ is not a constant, this requires $c_k = d_k$, *so that*

$$\frac{f_2(x)}{\rho_2} - \frac{f_3(x)}{\rho_1},$$

when expanded in the vicinity of a_k *contains no term of the form* $(x - a_k)^{-1}$. If $\rho_1 \rho_2 = \gamma =$ const., the only thing that follows is that $(c_k - d_k)\gamma$ must be an integer. However, in that case, we will consider equation (15) in place of the third equation (14), i. e., we require $f(x) = \log f_1(x)$ to be also a uniform func-

tion of x and then conclude again that $c_k = d_k$, unless $\rho_1 \rho_2 = 0$. Suppose that $\rho_1 \rho_2 = 0$. Then not both ρ_1 and ρ_2 vanish, for we have assumed $\rho_1 \neq \rho_2$. Suppose $\rho_2 = 0$ and $\rho_1 \neq 0$. Then ϑ_1 reduces to Y_1, ϑ_3 retains its form, but ϑ_2 loses its meaning. Instead of (14), we consider the system

$$\vartheta_3 = \frac{Y_2'}{Y_2} - \frac{\rho_1'}{\rho_1} \log Y_2 = f_3(x), \quad \vartheta_1 = f_1(x),$$

from which the validity of the theorem which we are attempting to prove follows easily for this special case.

In every case, then, Y_1, Y_2 are functions uniform except for $x = a_1, \ldots . a_m$, while by a circuit around a_i, (Y_1, Y_2) are transformed into $e^{\rho_2 t_k} Y_1$, $e^{\rho_1 t_k} Y_2$, t_k being a constant depending upon the coefficients of $f_2(x)$ and $f_3(x)$, namely, being equal to $c_k = d_k$, which can, moreover, be chosen at will. *Thus it is seen that functions* y_1, y_2 *exist which undergo arbitrary substitutions* A_i *of our group, when* x *makes a circuit around arbitrarily assigned points* a_i *in the plane.* In general, the point $x = \infty$ will also occur as branch point with the substitution A_{m+1}, so that $A_1 A_2 \ldots . A_m A_{m+1} = 1$. If, however, $A_1, \ldots . A_m$ are chosen, so that $A_1 \ldots . A_m = 1$, y_1, y_2 will be uniform at infinity. It is interesting to note that if ρ_1, ρ_2, f_2, f_3 are rational functions of x, $f_1(x)$ is necessarily a transcendental function or a constant.

Our group is algebraic if $\frac{\rho_2}{\rho_1}$ is a rational number, for then ϑ_1 is algebraic. This is essentially a theorem of Maurer's.

If we put $\lambda_1 = -\lambda_2 = i$, $\rho_1 = -\rho_2$, we get the group of rotations with the invariant $y_1^2 + y_2^2$. If the condition $\rho_1 = -\rho_2$ alone is upheld, we obtain a group with a homogeneous quadratic invariant.

§2.—*One-parameter Groups. Case II.*

Now let $\rho_1 = \rho_2 = \rho$ and $\lambda_1 = \lambda_2 = \lambda$, so that

$$D = (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 0, \tag{1}$$

and hence

$$\rho = \tfrac{1}{2}(\psi_{11} + \psi_{22}), \quad \lambda = -\frac{\psi_{22} - \psi_{11}}{2\psi_{21}} = + \frac{2\psi_{12}}{\psi_{22} - \psi_{11}}. \tag{2}$$

The finite equations of the group are

$$\left.\begin{aligned} \eta_1 &= (1+\psi_{21}\lambda t)\,e^{\rho t} y_1 - \lambda^2 \psi_{21}\, t e^{\rho t} y_2, \\ \eta_2 &= \psi_{21}\, t e^{\rho t} y_1 + e^{\rho t}(1-\lambda\psi_{21} t)\, y_2, \end{aligned}\right\} \tag{3}$$

or in a more symmetrical but less convenient form:

$$\left.\begin{aligned} \eta_1 &= [1-\tfrac{1}{2}(\psi_{22}-\psi_{11})\,t]\, e^{\rho t} y_1 + \tfrac{1}{2}\lambda(\psi_{22}-\psi_{11})\, t e^{\rho t} y_2, \\ \eta_2 &= -\frac{\psi_{22}-\psi_{11}}{2\lambda}\, t e^{\rho t} y_1 + [1+\tfrac{1}{2}(\psi_{22}-\psi_{11})\,t]\, e^{\rho t} y_2. \end{aligned}\right\} \tag{3a}$$

The invariants are easily found. We have from (3)

$$(\eta_1-\lambda\eta_2) = e^{\rho t}(y_1-\lambda y_2) \tag{4}$$

and $$\frac{\eta_2}{\eta_1-\lambda\eta_2} = \frac{\psi_{21}\, t y_1 + (1-\lambda\psi_{21} t)\, y_2}{y_1-\lambda y_2} = \frac{\psi_{21}\, t\,(y_1-\lambda y_2)+y_2}{y_1-\lambda y_2},$$

or $$\frac{\eta_2}{\eta_1-\lambda\eta_2} = \frac{y_2}{y_1-\lambda y_2} + \psi_{21} t. \tag{5}$$

Thus the expression

$$\vartheta_1 = (y_1-\lambda y_2)^{\psi_{21}}\, e^{-\rho\frac{y_2}{y_1-\lambda y_2}} \tag{6}$$

is an absolute invariant of our group, which might also have been found by integration.

The differential invariants of the first order can easily be formed from (4) and (5). They are

$$\left.\begin{aligned} \vartheta_2 &= \frac{d}{dx}\left[\frac{1}{\rho}\log(y_1-\lambda y_2)\right], \\ \vartheta_3 &= \frac{d}{dx}\left[\frac{1}{\psi_{21}}\;\frac{y_2}{y_1-\lambda y_2}\right], \end{aligned}\right\} \tag{7}$$

and the relation between ϑ_2, ϑ_3 and $\frac{d\vartheta_1}{dx}$ is

$$\frac{d}{dx}\;\frac{\log\vartheta_1}{\rho\psi_{21}} = \vartheta_2 - \vartheta_3, \tag{8}$$

or $$\frac{1}{\vartheta_1}\;\frac{d\vartheta_1}{dx} = \frac{1}{\rho\psi_{21}}\;\frac{d(\rho\psi_{21})}{dx}\log\vartheta_1 - \rho\psi_{21}(\vartheta_2-\vartheta_3). \tag{8a}$$

These expressions are not valid when $\rho = 0$ or $\psi_{21} = 0$. ρ and ψ_{21} cannot both vanish, for then the group would contain only the identical transformation. If $\rho = 0$, $\psi_{21} \neq 0$, we have the independent invariants

$$y_1 - \lambda y_2, \quad y_1' - \lambda y_2' - \lambda' y_2, \quad \frac{d}{dx}\left(\frac{1}{\psi_{21}}\,\frac{y_2}{y_1 - \lambda y_2}\right), \tag{9}$$

and if $\psi_{21} = 0$, $\rho \neq 0$,

$$\frac{y_2}{y_1 - \lambda y_2}, \quad \frac{d}{dx}\left(\frac{y_2}{y_1 - \lambda y_2}\right), \quad \frac{d}{dx}\left(\frac{1}{\rho}\log(y_1 - \lambda y_2)\right).^* \tag{9a}$$

In the general case, consider the system of differential equations

$$\vartheta_2 = f_2(x), \quad \vartheta_3 = f_3(x), \quad \vartheta_1 = f(x),$$

where $f_i(x)$ are uniform functions of x, and

$$\log f(x) = \rho\psi_{21}\int (f_2(x) - f_3(x))\, dx.$$

Applying our previous argument, word for word, we find that functions y_1, y_2 are thus determined, uniform, finite and continuous everywhere except for points $a_1, \ldots\ldots a_m$ arbitrarily chosen. Moreover, these points are branch-points of such a nature, that after a circuit of x around a_i, y_1, y_2 will have undergone an arbitrary linearoid substitution of our group. The special cases offer no exceptions.

It will be noticed that in both Case I and Case II the numerable subgroup generated by $A_1, \ldots\ldots A_m$ is such, that all of its substitutions are transformed into the canonical form of linear substitutions by the same transformation.

§3.—*Two-parameter Groups.*

Let the infinitesimal transformations be

$$\left.\begin{aligned} U_1 f &= (\phi_{11} y_1 + \phi_{12} y_2)\, q_1 + (\phi_{21} y_1 + \phi_{22} y_2)\, q_2 = \xi_1 q_1 + \xi_2 q_2, \\ U_2 f &= (\psi_{11} y_1 + \psi_{12} y_2)\, q_1 + (\psi_{21} y_1 + \psi_{22} y_2)\, q_2 = \eta_1 q_1 + \eta_2 q_2, \end{aligned}\right\} \tag{1}$$

where
$$q_i = \frac{\partial f}{\partial y_i}, \qquad (i = 1,\ 2)$$

* The case $\lambda = \infty$, in which y_2 is an invariant, causes no difficulty.

and where ϕ_{ik} and ψ_{ik} are uniform functions of x such that $U_1 f$ and $U_2 f$ generate a two-parameter linearoid group.

According to Lie's general theory, U_1, U_2 generate a group if, and only if,

$$(U_1, U_2) = c_1 U_1 + c_2 U_2, \tag{2}$$

where c_1 and c_2 are constants. Equation (2) is equivalent to the system of relations

$$\left.\begin{aligned} A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} &&= c_1\phi_{11} + c_2\psi_{11}, \\ A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} &&= c_1\phi_{12} + c_2\psi_{12}, \\ A_3 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} &&= c_1\phi_{21} + c_2\psi_{21}, \\ A_4 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} &&= c_1\phi_{22} + c_2\psi_{22}. \end{aligned}\right\} \tag{3}$$

Between the left members of these equations, the following relations hold:

$$\left.\begin{aligned} \phi_{21} A_2 + \phi_{12} A_3 + (\phi_{11} - \phi_{22}) A_1 &= 0, \\ \psi_{21} A_2 + \psi_{12} A_3 + (\psi_{11} - \psi_{22}) A_1 &= 0, \\ A_4 + \qquad\qquad\qquad A_1 &= 0. \end{aligned}\right\} \tag{4}$$

According to a general method of Lie, we consider the following cases in order:

I. $(U_1, U_2) = 0$, and no equation of the form $X_1 U_1 f + X_2 U_2 f = 0$, where X_1, X_2 are functions of x, y_1, y_2.

II. $(U_1, U_2) = 0, \qquad X_1 U_1 f + X_2 U_2 f = 0.$

In cases I and II, $c_1 = c_2 = 0$. If either of these is not zero, we can always put $c_1 = 1$, $c_2 = 0$. Thus, we have two more cases:

III. $(U_1, U_2) = U_1, \qquad X_1 U_1 f + X_2 U_2 f \neq 0.$

IV. $(U_1, U_2) = U_1, \qquad X_1 U_1 f + X_2 U_2 f = 0.$

We proceed to consider these cases in detail.

Case I.

According as certain of the quantities ϕ_{ik} and ψ_{ik} vanish, the infinitesimal transformations may assume different forms, which we shall write down, although

they may belong to the same *type* in Lie's terminology. We thus obtain not only all *types* but all *forms* of linearoid and linear groups.

First, suppose that neither ϕ_{21} nor ψ_{21} vanish identically. Then, since $A_1 = 0$, either $\phi_{12} = \psi_{12} = 0$, or neither ϕ_{12} nor ψ_{12} vanish. But the case $\phi_{12} = \psi_{12} = 0$ is identical except, as to notation, with $\phi_{21} = \psi_{21} = 0$, which will be discussed later. In Case I^a, then, take $\phi_{21}, \psi_{21}, \phi_{12}, \psi_{12}$ all different from zero. Equations $A_i = 0$ reduce to $A_1 = 0$, $A_3 = 0$, or

$$\frac{\phi_{12}}{\phi_{21}} = \frac{\psi_{12}}{\psi_{21}} = \omega_1(x), \quad \frac{\phi_{11} - \phi_{22}}{\phi_{21}} = \frac{\psi_{11} - \psi_{22}}{\psi_{21}} = \omega_2(x),$$

and, therefore, if we write

$$\phi_{22} = \phi_1, \quad \phi_{21} = \phi_1', \quad \psi_{22} = \phi_2, \quad \psi_{21} = \phi_2',$$

we have for form I^a,

$$\mathrm{I}^a. \quad U_i f = [(\phi_i + \omega_2 \phi_i') y_1 + \omega_1 \phi_i' y_2] q_1 + [\phi_i' y_1 + \phi_i y_2] q_2. \qquad (i = 1, 2).$$

Moreover, in order that the group may really be linearoid, i. e., in order that the finite equations of the group may have uniform coefficients, the characteristic equation of the general infinitesimal transformation must be reducible, i. e.,

$$\omega_2^2 + 4\omega_1 = \omega^2,$$

where ω is a uniform function of x. This condition is necessary and sufficient.

For form I^b, we have $\psi_{21} = 0$, $\phi_{21} \neq 0$,

$$\mathrm{I}^b. \quad \begin{cases} U_1 f = (\phi_{11} y_1 + \phi_{12} y_2) q_1 + (\phi_{21} y_1 + \phi_{22} y_2) q_2, \\ U_2 f = \qquad \psi(x)\,(y_1 q_1 + y_2 q_2), \end{cases}$$

and the group is linearoid if the one-parameter group $U_1 f$ is, i. e., if

$$D = (\phi_{11} + \phi_{22})^2 - 4(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) = \omega(x)^2,$$

where $\omega(x)$ is a uniform function of x.

In Case I^c, we have $\psi_{21} = 0$, $\phi_{21} = 0$. Only one equation, $A_i = 0$, is left, viz., $A_2 = 0$, or

$$(\psi_{11} - \psi_{22})\phi_{12} = (\phi_{11} - \phi_{22})\psi_{12}.$$

In Case I^{c1}, let $\psi_{12} = 0$, but $\phi_{12} \neq 0$. Then $\psi_{11} = \psi_{22}$ and we have U_1, U_2 of the same form as I^b, only that $\phi_{21} = 0$. If ϕ_{12} and ψ_{12} are both different from zero, we have

$$\frac{\phi_{11} - \phi_{22}}{\phi_{12}} = \frac{\psi_{11} - \psi_{22}}{\psi_{12}} = \omega,$$

and, therefore,

$$I^{c2}. \quad \begin{cases} U_1 f = [(\phi_{22} + \omega\phi_{12})\, y_1 + \phi_{12}\, y_2]\, q_1 + \phi_{22}\, y_2\, q_2, \\ U_2 f = [(\psi_{22} + \omega\psi_{12})\, y_1 + \psi_{12}\, y_2]\, q_1 + \psi_{22}\, y_2\, q_2. \end{cases}$$

If $\phi_{12} = \psi_{12} = 0$, we have

$$I^{c3}. \quad \begin{cases} U_1 f = \phi_{11}\, y_1\, q_1 + \phi_{22}\, y_2\, q_2, \\ U_2 f = \psi_{11}\, y_1\, q_1 + \psi_{22}\, y_2\, q_2. \end{cases}$$

Both groups are linearoid without further restrictions.

In all of these cases it is assumed that

$$\Delta = \xi_1 \eta_2 - \xi_2 \eta_1 \neq 0,$$

for otherwise there would be an equation of the form

$$X_1\, U_1 f + X_2\, U_2 f = 0.$$

Now, according to a general theorem of Lie's, if $U_1 f$ and $U_2 f$ generate a two-parameter group, Δ is a relative invariant. Let us put

$$(U_1,\, U_2) = c\, U_1, \quad c = 1 \text{ or } 0,$$

to include both cases. Then we find, employing a general method of Lie's (Transformationsgruppen, vol. I, p. 242),

$$U_1 \Delta = (\phi_{11} + \phi_{22})\, \Delta, \quad U_2 \Delta = (\psi_{11} + \psi_{22} - c)\, \Delta. \tag{5}$$

In our particular case, $c = 0$, and hence

$$U_1 \Delta = (\phi_{11} + \phi_{22})\, \Delta, \quad U_2 \Delta = (\psi_{11} + \psi_{22})\, \Delta. \tag{6}$$

We can always reduce the groups of Case I to the form

$$\frac{\partial f}{\partial \mathfrak{y}_1}, \quad \frac{\partial f}{\partial \mathfrak{y}_2},$$

i. e., to a group of two translations, by a transformation of coordinates. For, in order that this may be possible, it is necessary and sufficient to be able to choose

new variables $\mathfrak{y}_1$ and $\mathfrak{y}_2$, so that for any function $f(y_1, y_2)$,

$$\xi_1 \frac{\partial f}{\partial y_1} + \xi_2 \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial \mathfrak{y}_1}, \quad \eta_1 \frac{\partial f}{\partial y_1} + \eta_2 \frac{\partial f}{\partial y_2} = \frac{\partial f}{\partial \mathfrak{y}_2},$$

or, what amounts to the same thing, if we put $f = \mathfrak{y}_1$ and $f = \mathfrak{y}_2$ respectively, so that

$$\begin{cases} \xi_1 \dfrac{\partial \mathfrak{y}_1}{\partial y_1} + \xi_2 \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = 1, & \xi_1 \dfrac{\partial \mathfrak{y}_2}{\partial y_1} + \xi_2 \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = 0, \\ \eta_1 \dfrac{\partial \mathfrak{y}_1}{\partial y_1} + \eta_2 \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = 0, & \eta_1 \dfrac{\partial \mathfrak{y}_2}{\partial y_1} + \eta_2 \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = 1, \end{cases}$$

or that

$$\left.\begin{array}{ll} \dfrac{\partial \mathfrak{y}_1}{\partial y_1} = + \dfrac{\eta_2}{\Delta}, & \dfrac{\partial \mathfrak{y}_1}{\partial y_2} = - \dfrac{\eta_1}{\Delta}, \\ \dfrac{\partial \mathfrak{y}_2}{\partial y_1} = - \dfrac{\xi_2}{\Delta}, & \dfrac{\partial \mathfrak{y}_2}{\partial y_2} = + \dfrac{\xi_1}{\Delta}, \end{array}\right\} \tag{7}$$

Now, these equations can be integrated. For equations (6) are verified, and they are precisely the conditions of integrability for (7), and $\Delta \neq 0$. Or, what means the same thing, our groups are such as to make

$$d\mathfrak{y}_1 = \frac{\eta_2\, dy_1 - \eta_1\, dy_2}{\Delta}, \quad d\mathfrak{y}_2 = \frac{-\xi_2\, dy_1 + \xi_1\, dy_2}{\Delta} \tag{7a}$$

complete differentials, so that $\mathfrak{y}_1$ and $\mathfrak{y}_2$ are obtained by integrating (7a).

We thus obtain the canonical variables in every case. We find in Case I[a]

$$\left.\begin{array}{l} \Delta = (\phi_1 \phi_2' - \phi_2 \phi_1')(y_1^2 - \omega_1 y_2^2 - \omega_2 y_1 y_2) = (\phi_1 \phi_2' - \phi_2 \phi_1')(y_1 - \alpha y_2)(y_1 - \beta y_2), \\ \mathfrak{y}_1 = \quad \tfrac{1}{2} \phi_2' \log \Delta + \dfrac{\tfrac{1}{2} \phi_2' \omega_2 - \phi_2}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{1}{\beta - \alpha} \log \dfrac{y_1 - \beta y_2}{y_1 - \alpha y_2}, \\ \mathfrak{y}_2 = -\tfrac{1}{2} \phi_1' \log \Delta - \dfrac{\tfrac{1}{2} \phi_1' \omega_2 - \phi_1}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{1}{\beta - \alpha} \log \dfrac{y_1 - \beta y_2}{y_1 - \alpha y_2}, \end{array}\right\} \tag{8}$$

if $\alpha \neq \beta$. If $\alpha = \beta = \frac{1}{2} \omega_2$, we obtain

$$\left.\begin{array}{l} \mathfrak{y}_1 = \quad \tfrac{1}{2} \phi_2' \log \Delta - \dfrac{\tfrac{1}{2} \phi_2' \omega_2 - \phi_2}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{y_2}{y_1 - \tfrac{1}{2} \omega_2 y_2}, \\ \mathfrak{y}_2 = -\tfrac{1}{2} \phi_1' \log \Delta + \dfrac{\tfrac{1}{2} \phi_1' \omega_2 - \phi_1}{\phi_1 \phi_2' - \phi_2 \phi_1'} \, \dfrac{y_2}{y_1 - \tfrac{1}{2} \omega_2 y_2}. \end{array}\right\} \tag{8a}$$

In Case $\mathrm{I^b}$, we have

$$\Delta = \psi(x)\,[-\phi_{21}y_1^2 + \phi_{12}y_2^2 + (\phi_{11}-\phi_{22})\,y_1y_2] = \psi(x)(y_1-\alpha y_2)(y_1-\beta y_2),$$

and if $\alpha \neq \beta$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= -\frac{1}{\phi_{21}}\,\frac{1}{\beta-\alpha}\log\frac{y_1-\beta y_2}{y_1-\alpha y_2}, \\ \mathfrak{y}_2 &= -\frac{1}{2\psi(x)}\log\Delta + \frac{\phi_{11}-\phi_{22}}{2\psi(x)\,\phi_{21}}\log\frac{y_1-\beta y_2}{y_1-\alpha y_2}. \end{aligned}\right\} \tag{9}$$

If $\alpha = \beta = -\frac{1}{2}\,\dfrac{\phi_{11}-\phi_{22}}{\phi_{21}}$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{1}{\phi_{21}}\,\frac{y_2}{y_1-\alpha y_2}, \\ \mathfrak{y}_2 &= -\frac{1}{2\psi(x)}\log\Delta - \frac{\phi_{11}-\phi_{22}}{2\psi(x)\,\phi_{21}}\,\frac{y_2}{y_1-\alpha y_2}. \end{aligned}\right\} \tag{9a}$$

Group $\mathrm{I^{c1}}$ has the same form as $\mathrm{I^b}$, but (9) and (9a) are useless in this case since $\phi_{21}=0$. We find for $\mathrm{I^{c1}}$, if $\phi_{11}-\phi_{22}\neq 0$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= -\frac{1}{\phi_{11}-\phi_{22}}\log\left(\frac{y_1}{y_2}+\frac{\phi_{12}}{\phi_{11}-\phi_{22}}\right), \\ \mathfrak{y}_2 &= -\frac{1}{2\psi}\log\Delta - \frac{1}{2\psi}\log\left(\frac{y_1}{y_2}+\frac{\phi_{12}}{\phi_{11}-\phi_{22}}\right), \end{aligned}\right\} \tag{10}$$

and if $\phi_{11}-\phi_{22}=0$,

$$\mathfrak{y}_1 = -\frac{1}{\phi_{12}}\,\frac{y_1}{y_2}, \quad \mathfrak{y}_2 = -\frac{1}{2\psi}\log(\phi_{12}\psi y_2^2). \tag{10a}$$

In Case $\mathrm{I^{c2}}$, we find

$$\Delta = (\phi_{12}\psi_{22}-\phi_{22}\psi_{12})\,y_2(\omega y_1+y_2).$$

If $\omega \neq 0$, we have

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{\psi_{22}}{\omega(\phi_{12}\psi_{22}-\phi_{22}\psi_{12})}\log\left(\frac{y_1}{y_2}+\frac{1}{\omega}\right) - \frac{\psi_{12}}{\phi_{12}\psi_{22}-\phi_{22}\psi_{12}}\log y_2, \\ \mathfrak{y}_2 &= -\frac{\phi_{22}}{\omega(\phi_{12}\psi_{22}-\phi_{22}\psi_{12})}\log\left(\frac{y_1}{y_2}+\frac{1}{\omega}\right) + \frac{\phi_{12}}{\phi_{12}\psi_{22}-\phi_{22}\psi_{12}}\log y_2, \end{aligned}\right\} \tag{11}$$

and if $\omega = 0$,

$$\left.\begin{aligned} \mathfrak{y}_1 &= \frac{1}{\phi_{12}\psi_{22} - \phi_{22}\psi_{12}}\left(\quad \psi_{22}\frac{y_1}{y_2} - \psi_{12}\log y_2\right), \\ \mathfrak{y}_2 &= \frac{1}{\phi_{12}\psi_{22} - \phi_{22}\psi_{12}}\left(-\phi_{22}\frac{y_1}{y_2} + \phi_{12}\log y_2\right). \end{aligned}\right\} \tag{11a}$$

In case I^{c3}, we have

$$\left.\begin{aligned} \Delta &= (\phi_{11}\psi_{22} - \phi_{22}\psi_{11})\, y_1 y_2, \\ \mathfrak{y}_1 &= \frac{1}{\phi_{11}\psi_{22} - \phi_{22}\psi_{11}}\quad(\psi_{22}\log y_1 - \psi_{11}\log y_2), \\ \mathfrak{y}_2 &= \frac{1}{\phi_{11}\psi_{22} - \phi_{22}\psi_{11}}(-\phi_{22}\log y_1 + \phi_{11}\log y_2). \end{aligned}\right\} \tag{12}$$

If the groups are linearoid, in all cases where the determinant Δ has been separated into its factors $y_1 - \alpha y_2$ and $y_1 - \beta y_2$, α and β are uniform functions of x.

In every case $\frac{d\mathfrak{y}_1}{dx}$ and $\frac{d\mathfrak{y}_n}{dx}$ are differential invariants of the first order. The differential equations

$$\frac{d\mathfrak{y}_i}{dx} = r_i(x) \qquad (i = 1,\ 2) \tag{13}$$

are linearoid, belonging to the given two-parameter group. It is easy to see that $r_i(x)$ may be chosen as rational functions of x in such a way that the functions y_1, y_2, defined as solutions of (13), are uniform everywhere except in the vicinity of certain arbitrarily assigned points $a_1, \ldots a_m$, and that when x makes a circuit around a_i, y_1, y_2 undergo an arbitrary substitution A_i of the group.

These differential equations can also be obtained in another way. We have seen that Δ is a relative invariant. Factor Δ. It is found that each factor of Δ is also a relative invariant, which shows that the logarithm of each factor verifies a non-homogeneous linear differential equation. But this method fails if Δ is a perfect square, since it furnishes only one differential equation in that case.

We see again that the numerable subgroup generated by $A_1, \ldots A_m$ is such that all of its substitutions are reduced to the canonical form by the same transformation.

Case II.

In Case II, we have $(U_1, U_2)=0$, and besides a relation $X_1 U_1 f + X_2 U_2 f = 0$, so that $\Delta = 0$. We shall show that in this case $X_1 : X_2$ is a function of x alone. Generally, since

$$X_1 U_1 f + X_2 U_2 f = 0,$$

we can put

$$U_1 = \rho_1 U, \quad U_2 = \rho_2 U,$$

where U is an infinitesimal transformation, which we will call of degree λ if its coefficients contain y_1, y_2 in the λ^{th} power. Now, either U is of degree 1, and ρ_1 and ρ_2 are of degree 0, or U is of degree zero, and ρ_1 and ρ_2 of degree 1. In the first case we have at once $(U_1, U_2) = 0$, $\frac{X_1}{X_2}$ = function of x. In the second case, let

$$Uf = \lambda q_1 + \mu q_2, \quad \rho_1 = \phi_1 y_1 + \phi_2 y_2, \quad \rho_2 = \psi_1 y_1 + \psi_2 y_2.$$

Then we find

$$(U_1, U_2) = (\phi_1 \psi_2 - \phi_2 \psi_1)(\mu y_1 - \lambda y_2)\, Uf.$$

In order that this may be zero, we must have $\phi_1 \psi_2 - \phi_2 \psi_1 = 0$, i. e., $\rho_1 = \rho \times$ function of $x = \rho_2 . \rho(x)$ or $U_1 f = \rho_2(x)\, U_2 f$.

We have, therefore, shown that, in Case II, the infinitesimal transformations always have the form

$$\text{II.} \quad U_1 f, \quad X(x)\, U_1 f.$$

We can also, at this point, set down the form for Case IV. In that case, $(U_1, U_2) = U_1$, and only the second case, viz., that U is of degree zero, and $\rho_1 \rho_2$ of degree 1, can occur. The condition $(U_1, U_2) = U_1$ gives

$$\mu = \frac{\phi_1}{\phi_1 \psi_2 - \phi_2 \psi_1}, \quad \lambda = -\frac{\phi_2}{\phi_1 \psi_2 - \phi_2 \psi_1},$$

and hence

$$\text{IV.} \quad \begin{cases} U_1 f = \dfrac{\phi_1 y_1 + \phi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1}(-\phi_2 q_1 + \phi_1 q_2), \\[2ex] U_2 f = \dfrac{\psi_1 y_1 + \psi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1}(-\phi_2 q_1 + \phi_1 q_2). \end{cases}$$

Let us return to Case II. Suppose, first, that the characteristic equation of $U_1 f$ has unequal roots. Put, as in §1,

$$\mathfrak{y}_1 = Y_1^{\rho_1} Y_2^{-\rho_2}, \quad \mathfrak{y}_2 = \frac{1}{\rho_2} \log Y_1,$$

these being the canonical variables for the one-parameter group generated by $U_1 f$. Then $\mathfrak{y}_1$ is an invariant for the two-parameter group, and $\mathfrak{y}_2$ is transformed into

$$\bar{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_1 + c_2 X$$

by the most general transformation of the two-parameter group, c_1, c_2 being arbitrary constants. We obtain, therefore, the following differential invariant of the second order:

$$\frac{d}{dx} \frac{\frac{d\mathfrak{y}_2}{dx}}{\frac{dX}{dx}}.$$

The typical system of linearoid differential equations for this group is, therefore,

$$\frac{d}{dx} \frac{\frac{d\mathfrak{y}_2}{dx}}{\frac{dX}{dx}} = r(x), \quad \mathfrak{y}_1 = Y_1^{\rho_1} Y_2^{-\rho_2} = s(x),$$

If $\rho_1 = \rho_2$, we must put for $\mathfrak{y}_1$ and $\mathfrak{y}_2$ the expressions of §2.

Let $a_1, \ldots a_m$ be the poles of $r(x)$. Let a represent any one of them, and suppose that in the vicinity of $x = a$, we have

$$r(x) = \ldots + \frac{c_2}{(x-a)^2} + \frac{c_1}{x-a} + c_0 + \ldots$$

Then

$$\frac{d\mathfrak{y}_2}{dx} = \frac{dX}{dx}\left[\ldots - \frac{c_2}{x-a} c_1 \log(x-a) + c_0' + c_0(x-a) + \ldots\right]$$

$$= c_1 X' \log(x-a) + \ldots + \frac{e_1}{x-a} + e_0 + \ldots,$$

since $X(x)$ is also a uniform function of x. Moreover, c_1 and e_1 can be regarded

as absolutely arbitrary. Integration gives

$$\mathfrak{y}_2 = e_1 \log(x-a) + c_1\left[X\log(x-a) - \int X\frac{dx}{x-a}\right] + P(x-a) + P_1\left(\frac{1}{x-a}\right),$$

or if

$$\frac{X}{x-a} = \cdots\cdots \frac{l_1}{x-a} + l_0 + \cdots\cdots,$$

$$\mathfrak{y}_2 = (e_1 - c_1 l_1)\log(x-a) + c_1 X\log(x-a) + Q(x-a) + Q_1\left(\frac{1}{x-a}\right).$$

This shows at once that e_1 and c_1 can be chosen so that $\mathfrak{y}_2$ shall undergo an arbitrary substitution of the form

$$\overline{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_1 + Xc_2,$$

when x makes a circuit around a. The proof holds for any number of points a_k by resolving $r(x)$ into partial fractions. Correspondingly, y_1, y_2 undergo arbitrary linearoid substitutions of our group when x makes circuits about the arbitrarily selected points a_i.

Case III.

This case is characterized by the relations

$$(U_1, U_2) = U_1, \quad X_1 U_1 f + X_2 U_2 f \neq 0,$$

so that $c_1 = 1$, $c_2 = 0$.

Regarding equations (3) of this paragraph as equations for determining ϕ_{ik} as functions of ψ_{ik}, we can write them as follows:

$$\left.\begin{array}{lllll} -\phi_{11} & -\psi_{21}\phi_{12} & +\psi_{12}\phi_{21} & & =0, \\ -\psi_{12}\phi_{11} & +(\psi_{11}-\psi_{22}-1)\phi_{12} & & +\psi_{12}\phi_{22} & =0, \\ +\psi_{21}\phi_{11} & & +(\psi_{22}-\psi_{11}-1)\phi_{21} & -\psi_{21}\phi_{22} & =0, \\ & +\psi_{21}\phi_{12} & -\psi_{12}\phi_{21} & -\phi_{22} & =0. \end{array}\right\} \quad (14)$$

The determinant of these equations, which is

$$1 - (\psi_{11} - \psi_{22})^2 - 4\psi_{21}\psi_{12},$$

must vanish, i. e., we must have

$$D_2 = (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 1, \quad (15)$$

i. e., since D_2 is the discriminant of the characteristic equation of the infinitesimal transformation $U_2 f$, the roots of this equation must be distinct, and their difference must be equal to unity.

We obtain, further, by adding the first and last of equations (14).

$$\phi_{11} + \phi_{22} = 0. \tag{16}$$

Further, we notice that from the first relation (4), using equations (3), putting $c_1 = 1$, $c_2 = 0$, we obtain

$$2\phi_{12}\phi_{21} + (\phi_{11} - \phi_{22})\phi_{11} = 0,$$

or, according to (16),

$$\phi_{12}\phi_{21} + \phi_{11}^2 = 0. \tag{17}$$

But the discriminant of the characteristic equation of $U_1 f$ is

$$D_1 = (\phi_{11} - \phi_{22})^2 + 4\phi_{12}\phi_{21} = 4\left[\phi_{11}^2 + \phi_{12}\phi_{21}\right],$$

and, therefore,

$$D_1 = 0,$$

i. e., the roots of that equation are equal. But more than that, they are zero, for this equation, or

$$\omega^2 - (\phi_{11} + \phi_{22})\,\omega + \phi_{11}\phi_{22} - \phi_{12}\phi_{21} = 0$$

reduces to $\omega^2 = 0$ if (16) and (17) are fulfilled.

The following theorem is, therefore, true: *If $U_1 f$, $U_2 f$ generate a linearoid two-parameter group, such that*

$$(U_1, U_2) = U_1,$$

then the characteristic equation of $U_1 f$ has both of its roots equal to zero, while the roots of the characteristic equation of $U_2 f$ are distinct, and the difference between them is unity.

The first part of this theorem, if applied to linear groups, is a special case of a general theorem of Killing's. The theorem can obviously be generalized.

From (14) we find

$$\phi_{12} = -\frac{2\psi_{12}\phi_{11}}{1 - (\psi_{11} - \psi_{22})}, \quad \phi_{21} = \frac{2\psi_{21}\phi_{11}}{1 + \psi_{11} - \psi_{22}}, \quad \phi_{22} = -\phi_{11},$$

if $1 \pm (\psi_{11} - \psi_{22}) \neq 0$. Call this Case III[a]. Since ϕ_{11} is still arbitrary, put it equal to $[1 - (\psi_{11} - \psi_{22})^2]\,\phi(x)$, where $\phi(x)$ is also an arbitrary function of x. We have then

$$\text{III}^{a}. \quad \left\{ \begin{aligned} U_1 f &= \phi(x)\,[\{1 - (\psi_{11} - \psi_{22})^2\}\,y_1 - 2\psi_{12}\,\{1 + (\psi_{11} - \psi_{22})\}\,y_2]\,q_1 \\ &\quad + \phi(x)\,[2\psi_{21}\,\{1 - (\psi_{11} - \psi_{22})\}\,y_1 - \{1 - (\psi_{11} - \psi_{22})^2\}\,y_2]\,q_2, \\ U_2 f &= (\psi_{11} y_1 + \psi_{12} y_2)\,q_1 + (\psi_{21} y_1 + \psi_{22} y_2)\,q_2 \\ &\qquad (\psi_{11} - \psi_{22})^2 + 4\psi_{12}\psi_{21} = 1. \end{aligned} \right.$$

Moreover, we find

$$\Delta = (\xi_1 \eta_2 - \xi_2 \eta_1) = \phi\psi_{21}\,(1 - \psi_{11} - \psi_{22})(1 - \psi_{11} + \psi_{22})[y_1 - \lambda y_2]^2,$$

$$\lambda = \frac{1 + \psi_{11} - \psi_{22}}{2\psi_{21}} = \frac{2\psi_{12}}{1 - (\psi_{11} - \psi_{22})}.$$

Since we must have in our case $\Delta \neq 0$, the case that

$$\psi_{11} + \psi_{22} = 1$$

is also excluded from III[a].

Now let $1 - (\psi_{11} - \psi_{22}) = 0$, $1 + \psi_{11} - \psi_{22} = 2$. Equations (14) in this case (III[b]) reduce to

$$\left\{ \begin{aligned} -\phi_{11} \quad - \psi_{21}\phi_{12} + \psi_{12}\phi_{21} \qquad\qquad &= 0, \\ -\psi_{12}\phi_{11} \qquad\qquad\qquad\qquad + \psi_{12}\phi_{22} &= 0, \\ +\psi_{12}\phi_{11} \qquad - \quad 2\phi_{21} - \psi_{21}\phi_{22} &= 0, \\ +\psi_{21}\phi_{12} - \psi_{12}\phi_{21} - \quad \phi_{22} &= 0, \end{aligned} \right.$$

$$D_2 - 1 = 4\psi_{12}\psi_{21} = 0.$$

We have subcases to consider. First, in Case III[b1], let $\psi_{12} \neq 0$. Then $\psi_{21} = 0$, and since generally we find from the above equations

$$\psi_{12}(\phi_{11} - \phi_{22}) = 0, \quad \phi_{11} + \phi_{22} = 0,$$

we have $\phi_{11} = \phi_{22} = 0$, and from the first equation also $\phi_{21} = 0$, while $\phi_{12} = \phi$ remains arbitrary. We have then

$$\text{III}^{b1}. \quad \left\{ \begin{aligned} U_1 f &= \phi(x)\,y_2 q_1, \\ U_2 f &= (\psi_{11} y_1 + \psi_{12} y_2)\,q_1 + (\psi_{11} - 1)\,y_2 q_2. \end{aligned} \right.$$

Moreover,

$$\Delta = \phi\,(\psi_{11} - 1)\,y_2^2,$$

so that ψ_{11} must be different from unity.

If $\psi_{12}=0$ and $\psi_{21}=0$, we need only put $\psi_{12}=0$ in III[b1]. Let $\psi_{12}=0$, $\psi_{21}\neq 0$. Then

$$\phi_{12}=-\frac{\phi_{11}}{\psi_{21}}, \quad \phi_{21}=\psi_{21}\phi_{11}, \quad \phi_{22}=-\phi_{11}.$$

Thus we obtain, putting $\phi_{11}=\phi$,

$$\text{III}^{b2}. \quad \begin{cases} U_1 f=\phi(x)\left(y_1-\dfrac{1}{\psi_{21}}y_2\right)[q_1+\psi_{21}q_2], \\ U_2 f=\psi_{11}y_1q_1+[\psi_{21}y_1+(\psi_{11}-1)y_2]q_2, \end{cases}$$
$$\Delta=\phi\,\frac{1-\psi_{11}}{\psi_{21}}(\psi_{21}y_1-y_2)^2,$$

the case $\psi_{11}=1$ being again excluded.

Now let $1+\psi_{11}-\psi_{22}=0$, $1-(\psi_{11}-\psi_{22})=2$. We find if $\psi_{12}\neq 0$,

$$\text{III}^{c1}. \quad \begin{cases} U_1 f= \phi(x)(y_1-\psi_{12}y_2)[q_1+\dfrac{1}{\psi_{12}}q_2], \\ U_2 f= (\psi_{11}y_1+\psi_{12}y_2)q_1+(1+\psi_{11})y_2q_2, \\ \Delta=-\dfrac{\psi_{11}}{\psi_{12}}(y_1-\psi_{12}y_2)^2, \end{cases}$$

and since $\Delta\neq 0$, we must have $\psi_{11}\neq 0$.

If $\psi_{12}=0$, we have

$$\text{III}^{c2}. \quad \begin{cases} U_1 f= \phi(x)y_1q_2, \\ U_2 f= \psi_{11}y_1q_1+[\psi_{21}y_1+(1+\psi_{11})y_2]q_2, \\ \Delta=-\phi(x)\psi_{11}y_1^2. \end{cases}$$

We have now obtained all of the different forms of groups belonging to Case III. In every case Δ is a relative invariant. Since Δ is a perfect square, we have thus a linear invariant for every form belonging to Case III.

By a transformation of coordinates, the groups of Case III may be reduced to the form

$$\frac{\partial f}{\partial \mathfrak{y}_2}, \quad \mathfrak{y}_1\frac{\partial f}{\partial \mathfrak{y}_1}+\mathfrak{y}_2\frac{\partial f}{\partial \mathfrak{y}_2}, \tag{19}$$

as may be shown just as in Lie's "Vorlesungen über Differentialgleichungen," pp. 421–422. The following is a direct proof of this, and furnishes, besides, the formulæ for the transformation.

In order that $U_1 f$, $U_2 f$ may assume the form (19), it is necessary and sufficient that $\mathfrak{y}_1$ and $\mathfrak{y}_2$ be determined, so that

$$U_1(\mathfrak{y}_1) = 0, \quad U_2(\mathfrak{y}_1) = \mathfrak{y}_1, \quad U_1(\mathfrak{y}_2) = 1, \quad U_2(\mathfrak{y}_2) = \mathfrak{y}_2. \tag{20}$$

Now we have §3, (5),

$$U_1(\Delta) = 0, \quad U_2(\Delta) = (\psi_{11} + \psi_{22} - 1)\Delta, \tag{21}$$

so that we may put

$$\mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}. \tag{22}$$

We know further, taking the case that $\phi_{21} \neq 0$, that

$$U_1\left(\frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2}\right) = 1, \quad \lambda = -\frac{\phi_{22} - \phi_{11}}{2\phi_{21}} = +\frac{2\phi_{12}}{\phi_{22} - \phi_{11}}, \tag{23}$$

for the roots of the characteristic equation of $U_1 f$ are equal. Therefore,

$$U_1\left(\mathfrak{y}_2 - \frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2}\right) = 0,$$

and therefore, since every solution of $U_1 f = 0$ is a function of x and Δ,

$$\mathfrak{y}_2 = \frac{1}{\phi_{21}}\,\frac{y_2}{y_1 - \lambda y_2} + \Phi(\Delta, x). \tag{24}$$

The further condition $U_2(\mathfrak{y}_2) = \mathfrak{y}_2$ easily furnishes the most general form possible for $\Phi(\Delta, x)$, which contains an arbitrary function of x. But this is unnecessary, for we easily find

$$U_2\left(\frac{y_2}{y_1 - \lambda y_2} + \psi_{21}\right) = U_2\left(\frac{y_2}{y_1 - \lambda y_2}\right) = \frac{y_2}{y_1 - \lambda y_2} + \psi_{21}.$$

We can, therefore, put

$$\left.\begin{aligned} \mathfrak{y}_1 &= \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}, \\ \mathfrak{y}_2 &= \frac{1}{\phi_{21}}\left[\frac{y_2}{y_1 - \lambda y_2} + \psi_{21}\right]. \end{aligned}\right\} \tag{25}$$

Thus we find in the different cases the following sets of canonical variables:

$$\mathrm{III}^{a}.\quad \begin{cases} \mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}, \\ \mathfrak{y}_2 = \dfrac{1}{2(1-\psi_{11}+\psi_{22})}\left[\dfrac{1}{\psi_{21}}\,\dfrac{y_2}{y_1-\lambda y_2}+1\right], \\ \lambda = \dfrac{1+\psi_{11}-\psi_{22}}{2\psi_{21}} = \dfrac{2\psi_{12}}{1-(\psi_{11}-\psi_{22})}; \end{cases}$$

$$\mathrm{III}^{b1}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2(\psi_{11}-1)}},\qquad \mathfrak{y}_2 = \frac{\psi_{12}}{\phi}\left(\frac{1}{\psi_{12}}\,\frac{y_1}{y_2}+1\right);$$

$$\mathrm{III}^{b2}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2(\psi_{11}-1)}},\qquad \mathfrak{y}_2 = \frac{1}{\phi}\left(\frac{1}{\psi_{21}}\,\frac{y_2}{y_1-\frac{1}{\psi_{21}}y_2}+1\right);$$

$$\mathrm{III}^{c1}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2\psi_{11}}},\qquad \mathfrak{y}_2 = \frac{\psi_{12}}{\phi}\,\frac{y_2}{y_1-\psi_{12}y_2};$$

$$\mathrm{III}^{c2}.\quad \mathfrak{y}_1 = \Delta^{\frac{1}{2\psi_{11}}},\qquad \mathfrak{y}_2 = \frac{\psi_{21}}{\phi}\left(\frac{1}{\psi_{21}}\,\frac{y_2}{y_1}+1\right).$$

All of the forms III are truly linearoid. For if we compute the discriminant of the characteristic equation belonging to the general infinitesimal transformation of the group

$$c_1 U_1 f + c_2 U_2 f,$$

we find it to be equal to c_2^2, so that the difference of the roots is c_2. This completes a former theorem, so that we now know that *if U_1, U_2 generated a two-parameter group for which $(U_1, U_2) = U_1$, then the characteristic equation belonging to $U_1 f$ has its roots both zero, that belonging to $U_2 f$ has its roots distinct and their difference equal to* 1, *and that belonging to $c_1 U_1 + c_2 U_2$ has its roots distinct if $c_2 \neq 0$, and their difference equal to c_2.*

This theorem can also be read as a purely algebraic one, about the roots of three equations,

$$|\phi_{ik} - \delta_{ik}\rho| = 0,\quad |\psi_{ik} - \delta_{ik}\rho| = 0,\quad |c_1\phi_{ik} + c_2\psi_{ik} - \delta_{ik}\rho| = 0,$$
$$(\delta_{ik} = 0,\quad i \neq k,\quad \delta_{ii} = 1),$$

if the relations (3) of this section for $c_1 = 1$, $c_2 = 0$ take place between ϕ_{ik} and ψ_{ik}, and is doubtless true generally.

The two-parameter group induced by $U_1 f$, $U_2 f$ on $\mathfrak{y}_1$, $\mathfrak{y}_2$ is

$$\overline{\mathfrak{y}}_1 = a\mathfrak{y}_1,\quad \overline{\mathfrak{y}}_2 = a\mathfrak{y}_2 + b, \tag{26}$$

so that we have the following independent differential invariants of the first order

$$\frac{1}{\mathfrak{y}_1}\frac{d\mathfrak{y}_1}{dx}, \quad \frac{1}{\mathfrak{y}_1}\frac{d\mathfrak{y}_2}{dx}. \tag{27}$$

It may further be shown that to every transformation of the form (26) corresponds a transformation of our group in y_1 and y_2, which is perfectly definite, if only the meaning of such expressions as

$$a^{\frac{\psi_{11}+\psi_{22}-1}{2}} = e^{\frac{\psi_{11}+\psi_{22}-1}{2}\log a}$$

is made clear, by choosing one among the infinity of values of $\log a$. If, then, we can show that functions $\mathfrak{y}_1$, $\mathfrak{y}_2$ exist which undergo arbitrary substitutions of the form (26) for circuits of x around arbitrary points a_k, it follows that y_1, y_2 undergo the corresponding substitutions of our group for the same circuits. If, instead of $\mathfrak{y}_1 = \Delta^{\frac{1}{\psi_{11}+\psi_{22}-1}}$, we take $\mathfrak{y}_1 = (y_1 - \lambda y_2)^{\frac{2}{\psi_{11}+\psi_{22}-1}}$, we can also avoid the introduction of such algebraic branch points as are obtained, for instance, by solving the equations for Case III[a], which contain the square root

$$\sqrt{\phi\,(1-\psi_{11}+\psi_{22})(1-\psi-\psi_{22})\,\psi_{21}}.$$

Everything, then, depends upon the proof that functions $\mathfrak{y}_1$, $\mathfrak{y}_2$ exist, which for circuits of x around a_k, undergo the substitutions

$$\overline{\mathfrak{y}}_1 = \alpha_k\,\mathfrak{y}_1, \quad \overline{\mathfrak{y}}_2 = \alpha_k\,\mathfrak{y}_2 + \beta_k, \qquad (k = 1, 2, \ldots . m).$$

If there are such functions, $r_1(x)$ and $r_2(x)$, in

$$\frac{1}{\mathfrak{y}_1}\frac{d\mathfrak{y}_1}{dx} = r_1(x), \quad \frac{1}{\mathfrak{y}_1}\frac{d\mathfrak{y}_2}{dx} = r_2(x),$$

will be uniform functions. If we have

$$r_1(x) = \frac{1}{2\pi i}\sum_{k=1}^{m} \log \alpha_k \frac{1}{(x-a_k)} + \ldots . ,$$

we obtain

$$\mathfrak{y}_1 = \prod_{k=1}^{m} (x-a_k)^{\frac{1}{2\pi i}\log \alpha_k}\, u(x),$$

where $u(x)$ is a uniform function, and $\mathfrak{y}_1$ has the required property.

We then have

$$\mathfrak{y}_2 = \int_{\lambda_0}^{x} \prod_{k=1}^{m} (x - a_k)^{\frac{1}{2\pi i}\log \alpha_k} v(x)\, dx,$$

where $v(x)$ is also a uniform function of x, and where, for convenience, we have taken a definite integral.

Assume that $v(x)$ has no zeros or poles coincident with any point a_k, and that the expansion of $v(x)$ in the vicinity of its poles b contains no term of the form $(x - b)^{-1}$. Then the value of $\mathfrak{y}_2$ is changed only by circuits around a_k. For such a circuit we have

$$\overline{\mathfrak{y}}_2 = \alpha_k \mathfrak{y}_2 + (1 - \alpha_k) \int_{x_0}^{a_k} \prod_{k=1}^{m} (x - a_k)^{\frac{1}{2\pi i}\log \alpha_k} v(x)\, dx,$$

provided that the definite integral on the right member is well defined, or an equivalent equation involving a loop integral if this is not the case. If we take

$$v(x) = c_0 + c_1 x + \dots + c_{m-1} x^{m-1},$$

$c_0, \dots c_{m-1}$ can be so chosen that we obtain

$$\overline{\mathfrak{y}}_2 = \alpha_k \overline{\mathfrak{y}}_2 + \beta_k,$$

α_k and β_k being arbitrary constants, with this limitation, however, that if $\alpha_k = 1$, $\beta_k = 0$, so *that if a_k is no branch point for $\mathfrak{y}_1$, it is not for $\mathfrak{y}_2$.*

Case IV.

$$(U_1, U_2) = U_1, \quad X_1 U_1 + X_2 U_2 = 0.$$

We have already found for this case

$$\begin{cases} U_1 f = \dfrac{\phi_1 y_1 + \phi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1} (-\phi_2 q_1 + \phi_1 q_2) = \rho_1 U, \\ U_2 f = \dfrac{\psi_1 y_1 + \psi_2 y_2}{\phi_1 \psi_2 - \phi_2 \psi_1} (-\phi_2 q_1 + \phi_1 q_2) = \rho_2 U, \end{cases}$$

where

$$\phi_1 \psi_2 - \phi_2 \psi_1 \neq 0.$$

We have the absolute invariant

$$\mathfrak{y}_1 = \phi_1 y_1 + \phi_2 y_2 = \rho_1,$$

and find further

$$U_1\left(\frac{\rho_2}{\rho_1}\right)=1, \quad U_2\left(\frac{\rho_2}{\rho_1}\right)=\frac{\rho_2}{\rho_1}.$$

Putting, therefore,

$$\mathfrak{y}_1=\phi_1 y_1+\phi_2 y_2, \quad \mathfrak{y}_2=\frac{\psi_1 y_1+\psi_2 y_2}{\phi_1 y_1+\phi_2 y_2},$$

these variables are transformed by the two-parameter group

$$\overline{\mathfrak{y}}_1=\mathfrak{y}_1, \quad \overline{\mathfrak{y}}_2=\alpha\mathfrak{y}_2+\beta.$$

Our differential invariant is, therefore,

$$\frac{d\log}{dx}\frac{d\mathfrak{y}_2}{dx}.$$

The further discussion is similar to Case III.

§4.—*Three-parameter Groups.*

The first possible composition of a three-parameter group, assuming the group to be simple, is

$$I \quad (U_1,\ U_2)=U_1, \quad (U_1,\ U_3)=2U_2, \quad (U_2,\ U_3)=U_3. \tag{1}$$

These relations give at once

$$\left.\begin{aligned}&\phi_{11}+\phi_{22}=0, \quad \psi_{11}+\psi_{22}=0, \quad \chi_{11}+\chi_{22}=0,\\&D_1=(\phi_{11}-\phi_{22})^2+4\phi_{12}\phi_{21}=D_3=(\chi_{11}-\chi_{22})^2+4\chi_{12}\chi_{21}=0,\\&D_2=(\psi_{11}-\psi_{22})^2+4\psi_{12}\psi_{21}=1,\end{aligned}\right\} \tag{2}$$

where ϕ_{ik}, ψ_{ik}, χ_{ik} are the coefficients of U_1f, U_2f, U_3f respectively. By the methods of §3 we find

$$\left.\begin{aligned}&\phi_{11}=\phi, \quad \phi_{12}=-\frac{2\psi_{12}}{1-2\psi_{11}}\phi, \quad \phi_{21}=\frac{2\psi_{21}}{1+2\psi_{11}}\phi, \quad \phi_{22}=-\phi,\\&\chi_{11}=\chi, \quad \chi_{12}=\frac{2\psi_{12}}{1+2\psi_{11}}\chi, \quad \chi_{21}=-\frac{2\psi_{21}}{1-2\psi_{11}}\chi, \quad \chi_{22}=-\chi,\end{aligned}\right\} \tag{3}$$

where ϕ and χ are arbitrary functions of x. Equations (3) are the consequences of $(U_1,\ U_2)=U_1$, and $(U_2,\ U_3)=U_3$, provided that $\psi_{11}\neq\pm\frac{1}{2}$. Now, from

$(U_1,\ U_3) = 2U_2$, we obtain

$$\left.\begin{aligned} &\phi_{21}\chi_{12} - \phi_{12}\chi_{21} &&= 2\psi_{11}, \\ &\phi_{12}\chi_{11} - \phi_{11}\chi_{12} + \phi_{22}\chi_{12} - \phi_{12}\chi_{22} &&= 2\psi_{12}, \\ &\phi_{11}\chi_{21} - \phi_{21}\chi_{11} + \phi_{21}\chi_{22} - \phi_{22}\chi_{21} &&= 2\psi_{21}, \\ &\phi_{12}\chi_{21} - \phi_{21}\chi_{12} &&= 2\psi_{22}. \end{aligned}\right\} \tag{4}$$

Substituting the expressions (3), we obtain

$$-4\phi\chi\frac{\psi_{11}}{1-4\psi_{11}^2} = \psi_{11} = -\psi_{22},$$

$$\psi_{12} = -4\psi_{12}\frac{\phi\chi}{1-4\psi_{11}^2} = -\frac{\phi\chi}{\psi_{21}}, \quad \psi_{21} = -4\psi_{21}\frac{\phi\chi}{1-4\psi_{11}^2} = -\frac{\phi\chi}{\psi_{12}}.$$

or

$$\psi_{12}\psi_{21} = -\phi\chi. \tag{5}$$

as the only additional condition.

The roots of the characteristic equation belonging to $U_2 f$ are $+\frac{1}{2}$ and $-\frac{1}{2}$. *The roots of the characteristic equation belonging to the general infinitesimal transformation of the group,* $c_1 U_1 + c_2 U_2 + c_3$, *are found to be*

$$\rho_1 = -\rho_2 = \tfrac{1}{2}(c_2^2 - 2c_1 c_3). \tag{6}$$

The group is similar (ähnlich) to the special linear group by the transformation

$$\eta_1 = \frac{2}{1+2\psi_{11}}\left(y_1 - \frac{2\psi_{12}}{1-2\psi_{11}}y_2\right), \quad \eta_2 = y_1 + \frac{2\psi_{12}}{1+2\psi_{11}}y_2. \tag{7}$$

For we find

$$U_1(\eta_1) = 0, \quad U_1(\eta_2) = \eta_1; \quad U_3(\eta_1) = -\eta_2, \quad U_3(\eta_2) = 0;$$
$$U_2(\eta_1) = -\tfrac{1}{2}\eta_1, \quad U_2(\eta_2) = +\tfrac{1}{2}\eta_2,$$

so that if we introduce η_1, η_2 as new variables, the infinitesimal transformations become

$$\eta_1\bar{q}_2, \quad \tfrac{1}{2}(-\eta_1\bar{q}_1 + \eta_2\bar{q}_2), \quad -\eta_2\bar{q}_1,$$

which generate the special linear group.

The same is true, as is easily verified, if $\psi_{11} = \pm\frac{1}{2}$, the case not included in the above investigation.

The next possible composition is

$$(U_1,\ U_2) = 0, \quad (U_1,\ U_3) = U_1, \quad (U_2,\ U_3) = cU_2, \qquad (c \neq 0, \neq 1) \tag{8}$$

A rather long, but not uninteresting discussion shows that there are no linearoid groups of this composition.

Next we may have (Case II):

$$(U_1,\ U_2)=0,\quad (U_1,\ U_3)=U_1,\quad (U_2,\ U_3)=U_2. \tag{9}$$

We have at once

$$\left.\begin{aligned} U_1f&=\phi\left[\left(y_1-\frac{2\chi_{12}}{1-(\chi_{11}-\chi_{22})}\,y_2\right)q_1+\left(\frac{2\chi_{21}}{1+\chi_{11}-\chi_{22}}\,y_1-y_2\right)q_2\right],\\ U_2f&=\frac{\psi}{\phi}\,U_1f,\quad U_3f=(\chi_{11}y_1+\chi_{12}y_2)\,q_1+(\chi_{21}y_1+\chi_{22}y_2)\,q_2\\ &\qquad\qquad (\chi_{11}-\chi_{22})^2+4\chi_{12}\chi_{21}=1\end{aligned}\right\} \tag{10}$$

if $\chi_{11}-\chi_{22}\neq\pm 1$. If $\chi_{11}-\chi_{22}=\pm 1$, U_1f and U_2f change into forms such as as III[b1] .... III[c2] of §3, both being always of the same form, so that U_1f and U_2f have in this case always the same path-curves.

There is no group of the composition

$$(U_1,\ U_2)=0,\quad (U_1,\ U_3)=U_1,\quad (U_2,\ U_3)=U_1+U_2.$$

The next possibility is

$$(U_1,\ U_2)=0,\quad (U_1,\ U_3)=U_1,\quad (U_2,\ U_3)=0, \tag{11}$$

which gives

$$\left.\begin{aligned} U_1f&=\phi\left[\left(y_1-\frac{2\chi_{12}}{1-(\chi_{11}-\chi_{22})}\,y_2\right)q_1+\left(\frac{2\chi_{21}}{1+\chi_{11}-\chi_{22}}\,y_1-y_2\right)q_2\right],\\ U_2f&=\psi\,(y_1q_1+y_2q_2),\\ U_3f&=(\chi_{11}y_1+\chi_{12}y_2)\,q_1+(\chi_{21}y_1+\chi_{22}y_2)\,q_2,\end{aligned}\right\} \tag{12}$$

where

$$(\chi_{11}-\chi_{22})^2+4\chi_{12}\chi_{21}=1,\ \text{and}\ \chi_{11}-\chi_{22}\neq\pm 1. \tag{12a}$$

If $\chi_{11}-\chi_{22}=\pm 1$, U_1f assumes one of the forms III[b1] .... III[c3] of §3, but U_2f remains the same as in (12).

Next we may have

$$(U_1,\ U_2)=0,\quad (U_1,\ U_3)=0,\quad (U_2,\ U_3)=U_1, \tag{13}$$

from which we find either

$$\left.\begin{aligned} U_1f&=\phi\,(y_1q_1-y_2q_2),\\ U_2f&=\psi\,(y_1q_1+y_2q_2),\quad U_3f=\chi\,(y_1q_1+y_2q_2),\end{aligned}\right\} \tag{14}$$

or
$$\left.\begin{aligned} U_1 f &= \chi_{21}(\psi_{11} - \psi_{22})\, q_2 = \psi_{21}(\chi_{11} - \chi_{22})\, q_2, \\ U_2 f &= \psi_{11} y_1 q_1 + (\psi_{21} y_1 + \psi_{22} y_2)\, q_2, \quad \psi_{22} \neq 0, \\ U_3 f &= \chi_{11} y_1 q_1 + \left[\chi_{21} y_1 + \left(\chi_{11} - \frac{\phi_{21}}{\psi_{21}}\right) y_2\right] q_2, \end{aligned}\right\} \tag{14a}$$

or a similar group obtained by taking $\psi_{12} \neq 0$, $\psi_{21} = 0$.

There are, finally, a number of cases corresponding to

$$(U_1, U_2) = 0, \quad (U_1, U_3) = 0, \quad (U_2, U_3) = 0, \tag{15}$$

which it is necessary to write down.

The differential equations belonging to the several groups are easily found. In the first case the linearoid system is the transformed of a linear system of the form

$$\eta_i'' + p_1 \eta_i' + p_2 \eta_i = 0, \qquad (i = 1, 2), \tag{16}$$

by the equations (7), the transformation group of (16), in the sense of Picard, being the special linear homogeneous group.

In the next case, if we take for $\mathfrak{y}_1$, $\mathfrak{y}_2$ the canonical variables of Case III, §3, we have

$$U_1(\mathfrak{y}_1) = 0, \quad U_3(\mathfrak{y}_1) = \mathfrak{y}_1; \quad U_1(\mathfrak{y}_2) = 1, \quad U_3(\mathfrak{y}_2) = \mathfrak{y}_2,$$

and, therefore,

$$U_2(\mathfrak{y}_1) = 0, \quad U_2(\mathfrak{y}_2) = \lambda(x) = \frac{\psi}{\phi}.$$

The three-parameter group induced upon $\mathfrak{y}_1$, $\mathfrak{y}_2$ is, therefore,

$$\overline{\mathfrak{y}_1} = \alpha \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \alpha \mathfrak{y}_2 + \beta \lambda(x) + \gamma,$$

so that

$$\frac{1}{\mathfrak{y}_1} \frac{d\mathfrak{y}_1}{dx} = r_1(x), \quad \lambda'' \frac{\mathfrak{y}_2'}{\mathfrak{y}_1} - \lambda' \frac{\mathfrak{y}_2''}{\mathfrak{y}_1} = r_2(x),$$

are the differential equations, invariant under this group, λ', λ'', etc., denoting the first and second derivatives.

The function-theoretic nature of the functions $\mathfrak{y}_1$ and $\mathfrak{y}_2$ is easily investigated. As in §3, if a_k $(k = 1, 2, \ldots m)$ are the poles of $r_1(x)$, and $u(x)$ denotes a uniform function of x,

$$\mathfrak{y}_1 = \prod_{k=1}^{m} (x - a_k)^{r_k} u(x), \quad r_k = \frac{1}{2\pi i} \log \alpha_k,$$

so that, for a circuit around a_k, $\mathfrak{y}_1$ is multiplied by α_k.

Then

$$\mathfrak{y}_2' = \frac{\lambda'}{\lambda_0'}\left[c - \lambda_0' \int_{x_0}^{x} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\, dx\right],$$

$$\mathfrak{y}_2 = c' + \int_{x_0}^{x} \frac{\lambda'}{\lambda_0'}\left[c - \lambda_0' \int_{x_0}^{x} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\, dx\right] dx,$$

where $x = x_0$ is an ordinary point, and λ_0' denotes the value of λ' for $x = x_0$. c and c' are constants of integration.

Thus, for a circuit around a_k, we notice that $\mathfrak{y}_2'$ changes into

$$\overline{\mathfrak{y}_2'} = \alpha_k \mathfrak{y}_2' + \frac{(1-\alpha_k)}{\lambda_0'}\left[c - \lambda_0' \int_{x_0}^{a_k} \mathfrak{y}_1 \frac{r_2}{\lambda'^2}\, dx\right] \lambda'(x),$$

where, if $\alpha_k \neq 1$, the factor of $\lambda'(x)$ may be made to assume arbitrarily assigned values β_k by proper choice of the parameters still arbitrary in $u(x)$ and $r_2(x)$. And $\mathfrak{y}_2$ itself changes into

$$\overline{\mathfrak{y}_2} = \alpha_k \mathfrak{y}_2 + \beta_k \lambda(x) + \gamma_k,$$

where

$$\gamma_k = (1-\alpha_k)\int_{x_0}^{a_k} \mathfrak{y}_2' \, dx - \beta_k \lambda(a_k) + \beta^k \lambda(x_0),$$

by a proper choice of the infinitely many parameters still remaining in $u(x)$ and $r_2(x)$, can also be made to assume arbitrarily assigned values. This does not take into account possible exceptions, nor does it inform us as to the simplest functions of this kind. It suffices for our present purpose to note that, in general, functions exist with the arbitrary branch points a_k belonging to the substitutions

$$\overline{\mathfrak{y}_1} = \alpha_k \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \alpha_k \mathfrak{y}_2 + \beta_k \lambda(x) + \gamma_k,$$

where $\lambda(x)$ is a uniform function, and correspondingly there are functions y_1, y_2 with the same branch points and the corresponding linearoid substitutions.

Similar results are obtained from the groups of form (12), (14) and (14a). For groups of composition (15) suppose, first, that $U_i f\ (i = 1, 2, 3)$ have the same path-curves. Then, as we have seen in §3, Case II, $U_i f = \phi_i(x)\, Uf$. Calling $\mathfrak{y}_1$, $\mathfrak{y}_2$ the canonical variables of the group Uf, these quantities are transformed into

$$\overline{\mathfrak{y}_1} = \mathfrak{y}_1, \quad \overline{\mathfrak{y}_2} = \mathfrak{y}_2 + c_1 \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x),$$

so that $\mathfrak{y}_2$ verifies a non-homogeneous linear differential equation of the third

order, the corresponding homogeneous equation having ϕ_1, ϕ_2, ϕ_3 as a fundamental system.

If the path-curves of $U_i f$ are not all the same, suppose that $U_1 f$ and $U_2 f$ have distinct path-curves. Then $U_1 f$ and $U_2 f$ can be reduced, as in §3, to the canonical forms $\frac{\partial f}{\partial \mathfrak{y}_1}$, $\frac{\partial f}{\partial \mathfrak{y}_2}$, which gives for $U_3 f$ the form

$$\phi(x) \frac{\partial f}{\partial \mathfrak{y}_1} + \psi(x) \frac{\partial f}{\partial \mathfrak{y}_2},$$

ϕ and ψ being functions of x only. The group of $\mathfrak{y}_1\, \mathfrak{y}_2$ is then

$$\bar{\mathfrak{y}}_1 = \mathfrak{y}_1 + c_1 + c_3 \phi(x), \quad \bar{\mathfrak{y}}_2 = \mathfrak{y}_2 + c_2 + c_3 \psi(x),$$

so that both $\mathfrak{y}_1$ and $\mathfrak{y}_2$ verify non-homogeneous linear differential equations.

§5.—*Four and more Parameter Groups.*

Let $U_1 f \dots U_r f$ generate an r parameter linearoid group. If, in these infinitesimal transformations, we put $x = a$, the group becomes linear, and has $m \leqq 4$ parameters. Let $V_1 f, \dots V_r f$ be what becomes of $U_1 f, \dots U_r f$ when x is put equal to a.

Now, according to Lie's classification of binary linear groups, and as can be easily deduced from our results also, such a linear group is either the general linear group, or the special linear group, or, finally, a group conjugate under the general linear group with a group of the form

$$\eta_1 = \alpha y_1, \quad \eta_2 = \beta y_1 + \gamma y_2, \tag{1}$$

where, moreover, α, β, γ will depend upon a, if (1) represents the group generated by $V_1, \dots V_r$, and may have less than three essential parameters. But then the group generated by $U_1, \dots U_r$ will be of the same form with x in place of a. Such groups we will leave aside, as they do not present any peculiarity essentially different from those already studied.

We shall, therefore, suppose that $V_1, \dots V_r$ generate either the general or the special linear homogeneous group. In either case there will be three infinitesimal transformations, say V_1, V_2, V_3 which generate the special linear group among $V_1, \dots V_r$. Then U_1, U_2, U_3 must generate a three-parameter group with the same composition as the special linear group, which, moreover, we can suppose to be that group itself, only a linearoid transformation being necessary

for that purpose, as shown in §4. Thus we shall have

$$U_1 = y_1 q_1, \quad U_2 = \tfrac{1}{2}(-y_1 q_1 + y_2 q_2), \quad U_3 = -y_2 q_1,$$

and obviously all other linearoid infinitesimal transformations can be expressed in the form

$$U_k = \phi_1^{(k)} U_1 + \phi_2^{(k)} U_2 + \phi_3^{(k)} U_3 + \phi_4^{(k)}(y_1 q_1 + y_2 q_2), \quad (k = 4, 5, \ldots . r).$$

But (U_1, U_k) must be expressed as a linear function of $U_1, \ldots . U_r$ with constant coefficients. This requires $\phi_1^{(k)}, \phi_2^{(k)}, \phi_3^{(k)}$ to be constants. But $c_1 U_1 + c_2 U_2 + c_3 U_3$ already occurs in the group. Thus all other infinitesimal transformations of the group can be written in the form

$$U_k f = \phi_k(x)(y_1 q_1 + y_2 q_2), \quad (k = 4, 5, \ldots . r), \quad (2)$$

so that

$$(U_1, U_2) = U_1, \quad (U_1, U_3) = 2U_2, \quad (U_2, U_3) = U_3,$$
$$(U_1, U_k) = (U_2, U_k) = (U_3, U_k) = 0, \qquad (k = 4, 5, \ldots . r).$$

The finite equations of this group are

$$\eta_1 = \rho(\alpha y_1 + \beta y_2), \quad \eta_2 = \rho(\gamma y_1 + \delta y_2), \quad (3)$$

where

$$\rho = e^{\sum_{k=4}^{r} c_k \phi_k(x)}, \quad (4)$$

and the determinant

$$\alpha\delta - \beta\gamma = 1.$$

This latter restriction, however, is inessential, as $\phi_4(x)$ may, in particular, be equal to unity.

The corresponding differential equations are found as follows: The special linear group, i. e., the invariant subgroup U_1, U_2, U_3 has the differential invariants

$$u = y_1 y_2' - y_2 y_1', \quad v = y_1 y_2'' - y_2 y_1'', \quad w = y_1' y_2'' - y_2' y_1'',$$

and these are, therefore, transformed by $U_4, \ldots . U_r$ by an $r - 3$ parameter group, which we shall now investigate. We have, denoting by $U_k'' f$ the twice extended operators $U_k f$,

$$U_k'' f = \phi_k(y_1 q_1 + y_2 q_2) + (\phi_k y_1' + \phi_k' y_1) q_1' + (\phi_k y_2' + \phi_k' y_2) q_2'$$
$$+ (\phi_k y_1'' + 2\phi_k' y_1' + \phi_k'' y_1) q_1'' + (\phi_k y_2'' + 2\phi_k' y_2' + \phi_k'' y_2) q_2'',$$

where

$$q_i^{(k)} = \frac{\partial f}{\partial y_i^{(k)}}, \qquad (i, k = 1, 2).$$

We find

$$\left.\begin{aligned} U_k''(u) &= \quad 2\phi_k u, \\ U_k''(v) &= \quad 2\phi_k' u + 2\phi_k v, \\ U_k''(w) &= -\phi_k'' u + \phi_k' v + 2\phi_k w, \end{aligned}\right\} \tag{5}$$

so that u, v, w are transformed by an r-3 parameter group whose finite equations are

$$\left.\begin{aligned} \bar{u} &= e^{2\phi} u, \\ \bar{v} &= e^{2\phi}[2\phi' u + v], \\ \bar{w} &= e^{2\phi}[(\phi'^2 - \phi'') u + \phi' v + w], \end{aligned}\right\} \tag{5a}$$

where $\quad \phi = c_4\phi_4 + \ldots + c_r\phi_r, \ldots \phi'' = c_4\phi_4'' + \ldots + c_r\phi_r'';$ (5b)

the group is a ternary linearoid group.

We have, therefore,

$$\overline{\left(\frac{v}{u}\right)} = \frac{v}{u} + 2\phi',$$

$$\overline{\left(\frac{w}{u}\right)} = \frac{w}{u} + \phi'\frac{v}{u} + \phi'^2 - \phi'',$$

whence

$$\overline{\left(\frac{w}{u} - \frac{1}{4}\frac{v^2}{u^2}\right)} = \frac{w}{u} - \frac{1}{4}\frac{v^2}{u^2} - \phi'',$$

so that

$$\frac{w}{u} - \frac{1}{4}\frac{v^2}{u^2} + \frac{1}{2}\frac{d}{dx}\,\frac{v}{u} = I \tag{6}$$

is an absolute invariant under the group.

Now $\frac{v}{u}$ obviously verifies a non-homogeneous linear differential equation of order $r - 3$. The corresponding homogeneous equation has the fundamental system $\phi_4', \ldots \phi_r'$. Let $-\frac{v}{u} = p$, and let

$$\frac{d^{r-3}p}{dx^{r-3}} + s_1\frac{d^{r-4}p}{dx^{r-4}} + \ldots + s_{r-3}p = s \tag{7}$$

be the differential equation for p. Let $\frac{w}{u} = q$, then

$$q = I + \frac{1}{4}p^2 + \frac{1}{2}\frac{dp}{dx}. \tag{8}$$

But, according to the definition of p and q, y_1, y_2 are a fundamental system of the linear differential equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0. \tag{9}$$

Assuming y_1, y_2 to be uniform excepting multiformities expressible by the equations of our group, $s_1, \ldots\ldots s_{r-3}$, s, and I are uniform functions of x.

Our linearoid system consists, then, of equations (7), (8) and (9), or, instead of (9), the equivalent system

$$-\frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'} = p, \quad \frac{y_1' y_2'' - y_2' y_1''}{y_1 y_2' - y_2 y_1'} = q. \tag{9a}$$

This is of the third order, p and q being defined by (7) and (8), the general solu tions η_1, η_2 being given in terms of $y_1 y_2$ by the formulæ

$$\eta_1 = \alpha y_1 + \beta y_2, \quad \eta_2 = \gamma y_1 + \delta y_2, \quad \alpha\delta - \beta\gamma = 1.$$

The entire system is, therefore, of order r.

We have incidentally found a new proof for the well-known result that I is an invariant of (9) for the infinite group $\bar{y} = \lambda(x)\, y$.

We have seen that if $y_1 y_2$ are transformed by the group (3), the coefficients p and q of the differential equation (9) which they verify are transformed into

$$\left.\begin{aligned} \bar{p} &= p - 2\sum_{k=4}^{r} c_k \phi_k', \\ \bar{q} &= q + p\sum_{k=4}^{r} c_k \phi_k' + \Big(\sum_{k=4}^{r} c_k \phi_k'\Big)^2 - \sum_{k=4}^{r} c_k \phi_k''. \end{aligned}\right\} \tag{10}$$

The converse is also true, i. e., if p and q are transformed by (10), y_1, y_2 are transformed by (3). For, let us put

$$y = e^{-\frac{1}{2}\int p\,dx}\mathfrak{y},$$

then (9) becomes

$$\frac{d^2\mathfrak{y}}{dx^2} + I\mathfrak{y} = 0, \tag{11}$$

so that I being an invariant for (10), $\mathfrak{y}_1\, \mathfrak{y}_2$ are transformed by a linear transfor-

mation with constant coefficients. It has been shown by Klein and Poincaré that I may be chosen as a rational function of x, so that the solutions $\mathfrak{y}_1\, \mathfrak{y}_2$ have the arbitrary branch points $a_1, \ldots . a_m$ and undergo an arbitrary linear substitution A_k when x describes a closed path around a_k, provided that the roots of the fundamental equation belonging to A_k have the absolute value unity.

Moreover, it can be seen from the formula obtained by integrating (7), that s can be determined as rational function of x in such a way that for circuits around a_k, p undergoes arbitrary substitutions of form (10).

We thus have the result that functions y_1, y_2 *can be found with the arbitrary branch points* $a_1, \ldots . a_m$ *and the corresponding arbitrary linearoid substitutions of form* (3), *provided only that the roots of the equations of form*

$$\begin{vmatrix} \alpha - \omega, & \gamma \\ \beta \quad , & \delta - \omega \end{vmatrix} = 0$$

have their modulus equal to one.

These functions are such that the quotient $\frac{y_2}{y_1}$ undergoes projective substitutions with constant coefficients, and, therefore, verifies a differential equation of the form

$$\Delta\binom{\eta}{x} = u(x),$$

where $\Delta\binom{\eta}{x}$ is the Schwarzian derivative.

§6.—*Conclusion.*

In a former paper,* we have seen that the double-loop integrals which verify the hypergeometric differential equation, generalized by considering α, β, γ as uniform functions of x, are functions uniform except in the vicinity of $x = 0$, $1, \infty$, and that if x describes a circuit around one of these points, they undergo a linearoid substitution. These substitutions, we can now say, are not contained in any finite continuous linearoid group. The smallest continuous group containing them is an infinite group of linearoid transformations

$$\eta_i = \phi_{i1}(x)\, y_1 + \phi_{i2}(x)\, y_2, \qquad (i = 1, 2).$$

* This Journal, vol. XXI, No. 2.

The differential equations, if any, which these functions verify, cannot, therefore, be found by the methods of this paper. The functions of a similar nature obtained here are all very much simpler, and are, in fact, not essentially new. They are but peculiar combinations, as we now see, of solutions of linear differential equations.

The study of ternary linearoid groups and further generalizations, will be undertaken soon by Mr. F. E. Ross, a graduate student of the University of California.

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